B.A./B.Sc. II (SEMESTER-IV) / L.U. LUCKNOW

MATHEMATICS – PAPER-II (MECHANICS – DYNAMICS)

UNIT – IV (DYNAMICS – ROCKET MOTION)

E-CONTENT – LECTURE – NOTES-1

(PREPARED BY: Dr. DEEPAK KUMAR SRIVASTAVA
ASSOCIATE PROFESSOR, DEPTT. OF MATHS
B.S.N.V. P.G. COLLEGE, LUCKNOW)

---

§ **BASIC INFORMATION.** A rocket essentially consists of the mass of unburnt fuel (solid or liquid propellant), the mass of structure (casing, engine etc.) and the mass of payload (a warhead, a satellite to be put into orbit). It works on Newton's third Law. Through chemical reaction the fuel gets converted into a gas which escapes from the tail of the rocket at very high speed thereby producing a reactive force in the opposite direction. The main difference between a rocket engine and a jet engine is that while the latter is air breathing the former requires no atmosphere and actually works more efficiently in outer space. Since the fuel is being emitted from the rocket the problem is that of varying mass.

§ **EQUATION OF MOTION – FIRST STAGE ROCKET.**

Suppose that a body of mass $m(t)$ at time 't' is moving with velocity 'u'. Let mass be ejected from it at a rate α and with velocity v relative to the body. Thus, at time $t+\delta t$, the body will be having a mass $m-\alpha\delta t$ and moving with velocity $u+\delta u$.

On the other hand, the ejected mass $\alpha\delta t$ will be moving with velocity $u+v$. Thus change in momentum in time δt is equal to

$$(m-\alpha\delta t)(u+\delta u) + \alpha\delta t\,(u+v) - mu = m\delta u + \alpha v\delta t$$

(applying first order approximation)

$\therefore$ Rate of change of momentum is given by dividing above expression by δt following lim as $\delta t \to 0$

i.e;
$$\lim_{\delta t\to 0}\left[m\frac{\delta u}{\delta t} + \alpha v\right] = m\lim_{\delta t\to 0}\frac{\delta u}{\delta t} + \alpha v$$
$$= m\frac{du}{dt} + \alpha v.$$

If F be the external body force, from Newton's second Law, we have

Rate of change of momentum = external body force

$$\Rightarrow \quad m\frac{du}{dt} + \alpha v = F \qquad (I)$$

or
$$m\frac{du}{dt} = F - \alpha v$$

In case of rocket motion, α is positive but v (being opposite to u), is negative, hence resistive force $-\alpha v$, will be positive, i.e; it increases rocket velocity u.

As we are taking vertical motion, in present case, we have

$$F = -mg \quad , \quad m(t) = M + P - \alpha t ,$$

where M is the initial mass of rocket (casing, engine, unburnt fuel) and P is payload. If T is the time when all fuel has been exhausted beyond this time T the reactive force will cease to act, hence equation of motion (I) assumes following form

$$(M+P-\alpha t)\frac{du}{dt} = -(M+P-\alpha t)g - \alpha v . \quad (II)$$

$$\Rightarrow \quad \frac{du}{dt} = -g - v\left(\frac{\alpha}{M+P-\alpha t}\right)$$

On integrating this equation, we get the solution in the form by applying initial conditions

$t=0, \; u=u_0$ (initial velocity)

$$u = u_0 - v \log\left[1 - \frac{\alpha t}{M+P}\right] - gt , \quad 0 \le t \le T \quad (III)$$

For practical purposes, the time T is so small that the gravitational term is negligible in comparison to the second term at least at the all burnt stage $t=T$. Thus ultimate speed $u=V$ may be expressed as

$$V = v \log \frac{P+M}{P+M-\beta M} \quad \text{(IV)}$$

where $u_0 = 0$ & $\beta = \frac{\alpha t}{M}$.

If we take $P+M = M_1$ or M_i (the total initial mass)

& $P+M-\frac{\alpha T}{M}M = P+M-\beta M = M_1 - \alpha T = M_f$ (final mass at all fluel burnt stage)

Equation (IV) reduces to

$$V = v \log \frac{M_1}{M_f}$$

$$\Rightarrow \quad M_1 = M_f \, e^{V/v} \quad \text{(V)}$$

The relation given by equation (V) is known as "Tsiolkosky's formulae", which shows that to attain high values of V with some significant final mass M_f, the initial mass M_1 has to be enormously large.

§ SECOND STAGE ROCKET

Two stage rocket is designed in such a way that when the fuel in the first stage has burnt out, the first stage casing etc. falls off and the second stage engine takes over. With P as the payload, let us suppose that M_1 is the mass of the first stage and M_2 the mass of the second stage. We further assume that βM_1 is the initial mass of the fuel in the first stage and βM_2 in the second stage.

(on replacing $M \to M_2 + M_1$)

From equation (V), we get the speed V_1 attained by the rocket when the first stage is working as

$$V_1 = v \log \frac{P + M_2 + M_1}{P + M_2 + M_1'} = v \log \frac{P + M_2 + M_1}{P + M_2 + M_1 - \beta M_1}$$

where $M_1' = (1-\beta) M_1$ is the mass of the first stage without fuel.

The mass M_1' is cast off and taking the initial velocity as V_1, we can find final velocity at the end of stage two

$$V_2 = V_1 + v \log \frac{P + M_2}{P + M_2'}$$

$$= v \log \frac{P + M_2 + M_1}{P + M_2 + M_1'} + v \log \frac{P + M_2}{P + M_2'}$$

$$= v \log \frac{(P + M_2)(P + M_2 + M_1)}{(P + M_2 + M_1')(P + M_2')}$$

where $M_2' = (1-\beta) M_2$ is the mass of the second stage without fuel.

§ THIRD STAGE ROCKET.

In a three stage rocket, with obvious meaning of the symbols, the second stage rocket expression

$$V_2 = v \log \frac{(P+M_2)(P+M_2+M_1)}{(M_2'+P)(P+M_2+M_1')}$$

assumes the form

$$V_3 = v \log \frac{(P+M_3)(P+M_3+M_2)(P+M_3+M_2+M_1)}{(P+M_3')(P+M_3+M_2')(P+M_3+M_2+M_1')}$$

and this may be continued to an n-stage rocket.

NOTE. For projecting an artificial satellite in an orbit round the Earth, it may be shown that a three stage rocket is most suitable.

REFERENCES

1. LECTURE NOTES ON "MECHANICS"
By: SUNIL DATTA
PUBLICATION BY: BHARATA GANITA PARISHAD, LUCKNOW.